

# IRREDUCIBILITY OF SOME REPRESENTATIONS OF THE GROUPS OF SYMPLECTOMORPHISMS AND CONTACTOMORPHISMS

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**ABSTRACT.** We show the irreducibility of some unitary representations of the group of symplectomorphisms and the group of contactomorphisms.

## 1. INTRODUCTION

Let  $M$  be a smooth second-countable manifold. There exists a natural diffeomorphism-invariant measure class on  $M$ , consisting of measures having positive density with respect to the Lebesgue measure in every coordinate chart. We will refer to them simply as Lebesgue measures.

Let  $\mu$  be a Lebesgue measure on  $M$ . For a group  $G$  acting on  $M$  by diffeomorphisms we may consider a series  $\Pi_\mu^\theta$  of unitary representations on  $L^2(M, \mu)$  given by

$$(1.1) \quad \Pi_\mu^\theta(\gamma)f = f \circ \gamma^{-1} \left( \frac{d\gamma_*\mu}{d\mu} \right)^{1/2+i\theta},$$

where  $\theta \in \mathbb{R}$ .

If a measure  $\nu$  is equivalent to  $\mu$ , then the operator  $T: L^2(M, \mu) \rightarrow L^2(M, \nu)$  defined by

$$(1.2) \quad Tf = f \left( \frac{d\mu}{d\nu} \right)^{1/2+i\theta}$$

gives an isomorphism of representations  $\Pi_\mu^\theta$  and  $\Pi_\nu^\theta$ . In particular, if  $\mu$  is equivalent to a  $G$ -invariant measure, the representations  $\Pi_\mu^\theta$  are equivalent for all  $\theta \in \mathbb{R}$ .

For a diffeomorphism  $\phi: M \rightarrow M$  we define its support  $\text{supp } \phi$  as the closure of the set  $\{p \in M : \phi(p) \neq p\}$ . Compactly supported diffeomorphisms of  $M$  form a group  $\text{Diff}_c(M)$ . In [6] it was proved that for an infinite measure  $\mu$  the representation  $\Pi_\mu^0$  of the group  $\text{Diff}_c(M, \mu)$

of compactly supported, measure-preserving diffeomorphisms of  $M$  is irreducible. It follows that the representations  $\Pi_\mu^\theta$  of the groups  $\text{Diff}_c(M, \mu)$  and  $\text{Diff}_c(M)$  are irreducible for any  $\theta \in \mathbb{R}$ . The idea of the proof is to take two functions  $f, g \in L^2(M, \mu)$  and explicitly find a diffeomorphism  $\phi$  such that  $\langle f, \Pi_\mu^0(\phi)g \rangle \neq 0$ , thus showing that  $f$  and  $g$  cannot lie in two distinct orthogonal invariant subspaces.

Representations of various subgroups of the group of diffeomorphisms are also studied in [4].

The purpose of this note is to present an enhancement of the argument from [6], and apply it to classical groups of diffeomorphisms: the group of symplectomorphisms and the group of contactomorphisms.

## 2. CONVOLUTION ON THE HEISENBERG GROUP

On  $\mathbb{R}^n$  the following theorem holds (see Theorem 4.3.3 in [3] for a proof of a more general result):

**Theorem 2.1.** *If  $f, g \in L^1(\mathbb{R}^n)$  are compactly supported and nonzero, then  $f * g$  is nonzero.*

*Proof.* Let  $\hat{h}(\xi) = \int h(x)e^{-ix\xi} dx$  denote the Fourier transform of  $h \in L^1(\mathbb{R}^n)$ . Suppose that  $f * g = 0$ . As  $f$  and  $g$  are compactly supported, their Fourier transforms extend to entire functions. Since  $\hat{f}\hat{g} = \widehat{f * g} = 0$  on  $\mathbb{R}^n$ , it follows by holomorphicity that  $\hat{f}\hat{g} = 0$  on  $\mathbb{C}^n$ , and either  $\hat{f}$  or  $\hat{g}$  must vanish. This contradicts the assumption that  $f$  and  $g$  are nonzero.  $\square$

In this section we will prove an analogue of this theorem for square-integrable functions on the Heisenberg group.

**2.1. The Heisenberg group.** Let  $n$  be a positive integer. The multiplicative group of all matrices of the form

$$(2.1) \quad \begin{pmatrix} 1 & \bar{x}^T & z \\ 0 & I_n & \bar{y} \\ 0 & 0 & 1 \end{pmatrix},$$

where  $z \in \mathbb{R}$ ,  $\bar{x}, \bar{y} \in \mathbb{R}^n$ , and  $I_n$  denotes the  $n \times n$  identity matrix, is called the Heisenberg group  $H_n$ . It is a unimodular Lie group diffeomorphic with  $\mathbb{R}^{2n+1}$ , and its Haar measure is the  $(2n+1)$ -dimensional Lebesgue measure. We will identify  $H_n$  with  $\mathbb{R}^{2n+1}$  as manifolds. The convolution of functions  $f, g \in L^1(H_n)$  will be denoted  $f *_H g$ .

**2.2. Convolution of compactly supported functions on  $H_n$ .** Let  $f \in L^1(\mathbb{R})$ . Define

$$(2.2) \quad Tf(x) = \int_{-\infty}^x f(t) dt.$$

If  $f \in L^2(\mathbb{R})$  is supported in  $[a, b]$ , then it is integrable; furthermore, if  $\int f(t) dt = 0$ , then  $\text{supp } Tf \subseteq [a, b]$  and we may write

$$(2.3) \quad Tf(x) = \int f(t)K_{[a,b]}(t, x) dt,$$

where

$$(2.4) \quad K_{[a,b]}(t, x) = \begin{cases} 1 & \text{for } a \leq t \leq x \leq b, \\ 0 & \text{otherwise.} \end{cases}$$

Hence,  $Tf \in L^2(\mathbb{R})$  and  $\|Tf\|_2 \leq \|K_{[a,b]}\|_2 \|f\|_2$ , where  $\|K_{[a,b]}\|_2$  stands for the  $L^2$ -norm of  $K_{[a,b]} \in L^2(\mathbb{R}^2)$ . We may iterate the process of applying  $T$  to  $f$  as long as it yields a function integrating to 0. The next lemma shows that unless  $f = 0$ , this process terminates.

**Lemma 2.2.** *If  $f \in L^2(\mathbb{R})$  is nonzero and compactly supported, then there exists  $k \geq 0$  such that  $T^k f \in L^2(\mathbb{R})$  and  $\int T^k f(x) dx \neq 0$ .*

*Proof.* If there is no such  $k$ , then  $T^k f \in L^2(\mathbb{R})$  and  $\int T^k f(x) dx = 0$  for all  $k$ . Suppose this is the case. We may assume that  $\text{supp } f \subseteq [0, 1]$ , and replace  $T$  with a bounded operator of the form (2.3) with kernel  $K_{[0,1]}$ .

Since  $f$  is compactly supported,  $\hat{f}$  extends to an entire function on  $\mathbb{C}$ . We now have

$$(2.5) \quad \widehat{T^k f}(\xi) = (i\xi)^{-k} \hat{f}(\xi),$$

and by the Plancherel theorem

$$(2.6) \quad 4\pi^2 \|T^k f\|_2^2 = \|\widehat{T^k f}\|_2^2 \geq \int_{-1}^1 |\hat{f}(\xi)|^2 d\xi$$

But  $\|T\| \leq \|K_{[0,1]}\|_2 < 1$ , so the left-hand side of the above inequality can be made arbitrarily small. Therefore  $\hat{f} = 0$ , as it is an entire function vanishing on  $[-1, 1]$ . This contradicts the assumption that  $f$  is nonzero.  $\square$

Let  $f \in L^2(H_n)$  be compactly supported. Define  $Sf \in L^1(\mathbb{R}^{2n})$  by

$$(2.7) \quad Sf(\bar{x}, \bar{y}) = \int_{\mathbb{R}} f(\bar{x}, \bar{y}, z) dz.$$

If  $Sf = 0$ , we may also define  $Tf \in L^2(H_n)$  by

$$(2.8) \quad Tf(\bar{x}, \bar{y}, z) = \int_{-\infty}^z f(\bar{x}, \bar{y}, t) dt$$

The proof of the next lemma consists of a straightforward application of the Fubini theorem:

**Lemma 2.3.** *If  $f, g \in L^2(H_n)$  are compactly supported, then*

- (1)  $S(f *_H g) = Sf *_H Sg$ ,
- (2) *if  $Sf = 0$ , then  $(Tf) *_H g = T(f *_H g)$ ,*
- (3) *if  $Sg = 0$ , then  $f *_H (Tg) = T(f *_H g)$ .*

**Theorem 2.4.** *If  $f, g \in L^2(H_n)$  are compactly supported and nonzero, then  $f *_H g \neq 0$ .*

*Proof.* By Lemma 2.2 there exist minimal  $k$  and  $l$  such that  $ST^k f, ST^l g \in L^1(\mathbb{R}^{2n})$  are nonzero and compactly supported. From Lemma 2.3 and Theorem 2.1 we obtain

$$(2.9) \quad ST^{k+l}(f *_H g) = S(T^k f *_H T^l g) = ST^k f *_H ST^l g \neq 0,$$

which implies that  $f *_H g \neq 0$ . □

### 3. SYMPLECTIC MANIFOLDS

**3.1. Symplectic manifolds.** Let  $M$  be a symplectic manifold, that is a  $2n$ -dimensional manifold equipped with a nondegenerate closed 2-form. A symplectomorphism of  $(M, \omega)$  is a diffeomorphism  $\phi \in \text{Diff}(M)$  satisfying  $\phi^* \omega = \omega$ . The group of all compactly supported symplectomorphisms will be denoted by  $\text{Symp}_c(M, \omega)$ . Since  $\omega$  is nondegenerate,  $\omega^n$  defines a positive measure  $\mu$  on  $M$ , invariant under the action of  $\text{Symp}_c(M, \omega)$ .

A standard example of a symplectic manifold is  $\mathbb{R}^{2n}$  endowed with the symplectic form  $\omega_0 = \sum_{i=1}^n dx^i \wedge dy^i$ . It is a theorem of Darboux that any symplectic manifold is locally symplectomorphic to  $(\mathbb{R}^{2n}, \omega_0)$ :

**Theorem 3.1.** *For every  $p \in M$  there exists a chart  $\phi: U \rightarrow \mathbb{R}^{2n}$  centered at  $p$ , such that  $\omega|_U = \phi^* \omega_0$ .*

*Proof.* See [1], Theorem 8.1. □

The chart satisfying the conditions of Theorem 3.1 is called a Darboux chart. The pushforward of  $\mu$  through a Darboux chart is the standard Lebesgue measure, up to a constant factor.

The flow  $\text{Fl}_t^X$  of a complete vector field  $X \in \mathfrak{X}(M)$  consists of symplectomorphisms if and only if

$$(3.1) \quad \mathcal{L}_X \omega = 0.$$

There is an easy way to produce such vector fields. Namely, consider a compactly supported smooth function  $f \in C^\infty(M)$ . Since  $\omega$  is non-degenerate, there exists a unique vector field  $X_f \in \mathfrak{X}(M)$  such that  $X_f \lrcorner \omega = df$ , and it is not hard to show that this field satisfies (3.1).

For more information on symplectic manifolds see [1] and [5].

**3.2. The representation  $\Pi_\mu^0$  of  $\text{Symp}_c(M, \omega)$ .** As  $\mu$  is a  $\text{Symp}_c(M, \omega)$ -invariant measure, the only interesting representation is  $\Pi_\mu^0$ , taking the form

$$(3.2) \quad \Pi_\mu^0(\gamma)f = f \circ \gamma^{-1}.$$

Notice that the space of constant square-integrable functions is  $\Pi_\mu^0$ -invariant. It is nontrivial when  $\mu(M) < \infty$ . Let us denote its orthogonal complement by  $\mathcal{H}$ .

**Theorem 3.2.** *The representation  $\Pi_\mu^0$  of the group  $\text{Symp}_c(M, \omega)$  on the space  $\mathcal{H}$  is irreducible.*

**Lemma 3.3.** *Let  $p \in M$  and let  $\phi: U \rightarrow \mathbb{R}^{2n}$  be a Darboux chart centered at  $p$ . Then there exist  $r > 0$  and for every  $x \in B(0, 2r)$  a symplectomorphism  $\tau_x \in \text{Symp}_c(U, \omega|_U) \subseteq \text{Symp}_c(M, \omega)$  such that*

- (1)  $\overline{B(0, 3r)} \subseteq \phi[U]$ ,
- (2)  $\phi\tau_x\phi^{-1}(y) = y + x$  for all  $y \in B(0, r)$ .

*Proof.* Take  $r > 0$  satisfying (1) and a bump function  $h \in C^\infty(\mathbb{R}^{2n})$  supported in  $\phi[U]$  and equal to 1 on  $\overline{B(0, 3r)}$ . On  $\mathbb{R}^{2n}$  there exists a linear function  $f$  such that  $X_f = x$  is a constant field. Then  $X_{fh} = x$  on  $\overline{B(0, 3r)}$  and  $\text{supp } X_{fh} \subseteq \phi[U]$ . The desired symplectomorphism is  $\tau_x = \phi^{-1}\text{Fl}_1^{X_{fh}}\phi$ .  $\square$

By using a standard argument we obtain the following well-known corollary:

**Corollary 3.4.** *The action of  $\text{Symp}_c(M, \omega)$  on  $M$  is  $k$ -transitive for all  $k \geq 1$ .*

**Lemma 3.5.** *Let  $\phi: U \rightarrow \mathbb{R}^{2n}$  be a Darboux chart. Then for every nontrivial  $\Pi_\mu^0$ -invariant subspace  $\mathcal{H}_0$  of  $\mathcal{H}$ , there exists  $f \in \mathcal{H}_0$  such that  $f \neq 0$  and  $\text{supp } f \subseteq U$ .*

*Proof.* We may assume that  $0 \in U = \phi[U] \subseteq \mathbb{R}^{2n}$ . Let  $r > 0$  be as in Lemma 3.3. Take a nonzero  $g \in \mathcal{H}_0$ . The 2-transitivity of  $\text{Symp}_c(M, \omega)$  allows us to assume without loss of generality that there exists  $c \in \mathbb{R}$  such that the sets  $A = \{p \in B(0, r) : \text{Re } g(p) < c\}$

and  $B = \{p \in B(0, r) : \operatorname{Re} g(p) > c\}$  both have positive measure. By the Lebesgue density theorem there exist  $a \in A$  and  $b \in B$  with the property that  $A$  (resp.  $B$ ) has Lebesgue density 1 at  $a$  (resp.  $b$ ). Lemma 3.3 asserts the existence of a symplectomorphism  $\tau = \tau_{b-a}$  that takes  $a$  onto  $b$  and preserves the Lebesgue density on  $B(0, 3r)$ . The function  $f = g - \Pi_\mu^0(\tau)g \in \mathcal{H}_0$  then satisfies the conclusion of the lemma.  $\square$

*Proof of Theorem 3.2.* Suppose that  $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_0^\perp$  is a nontrivial decomposition into  $\Pi_\mu^0$ -invariant subspaces. Let  $\phi: U \rightarrow \mathbb{R}^{2n}$  be a Darboux chart, and let  $r > 0$  and  $\tau_x \in \operatorname{Symp}_c(M, \omega)$  be as in Lemma 3.3. Without loss of generality assume that  $U = \phi[U] \subseteq \mathbb{R}^{2n}$ . By Lemma 3.5 we may choose nonzero  $f \in \mathcal{H}_0$  and  $g \in \mathcal{H}_0^\perp$  supported in  $B(0, r)$ . We have

$$(3.3) \quad \langle f, \Pi_\mu^0(\tau_x)g \rangle = \int_{B(0, r)} f(y) \overline{g(\tau_x^{-1}(y))} dy = f * g^*(x),$$

where  $g^*(y) = \overline{g(-y)}$ . But from Theorem 2.1 we know that this is nonzero for some  $x \in \operatorname{supp} f * g^* \subseteq B(0, 2r)$ . We obtain a contradiction, since  $\Pi_\mu^0(\tau_x)g \in \mathcal{H}_0^\perp$ .  $\square$

#### 4. CONTACT MANIFOLDS

**4.1. Contact manifolds.** Let  $\dim M = 2n + 1$ . A contact form on  $M$  is a 1-form  $\alpha \in \Omega^1(M)$  such that  $\alpha \wedge (d\alpha)^n$  is a volume form. Consider a  $2n$ -dimensional distribution  $\xi \leq TM$ . There exists an open cover  $\mathcal{U} = \{U_i\}$  of  $M$ , such that for every  $U \in \mathcal{U}$  the restriction  $\xi|_U$  is the kernel of a 1-form  $\alpha_U \in \Omega^1(U)$ . If the forms  $\alpha_U$  are contact forms, we call  $(M, \xi)$  a contact manifold. Unless  $\xi$  is the kernel of a globally defined contact form, there is no distinguished measure on  $M$ .

Assume for the rest of this section that  $(M, \xi)$  is a contact manifold. A contactomorphism of  $(M, \xi)$  is a diffeomorphism  $\phi \in \operatorname{Diff}(M)$ , such that  $\phi_*\xi = \xi$ . The group of compactly supported contactomorphisms will be denoted by  $\operatorname{Cont}_c(M, \xi)$ .

An example of a contact manifold is the Heisenberg group  $H_n$  with the distribution  $\xi = \ker \alpha_0$ , where  $\alpha_0 = dz - \sum_i y^i dx^i$  is a right-invariant form on  $H_n$ .

There is an analogue of Darboux theorem for contact manifolds:

**Theorem 4.1.** *For every  $p \in M$  there exists a chart  $\phi: U \rightarrow H_n$  centered at  $p$ , such that  $\xi|_U = \ker \phi^*\alpha_0$ .*

*Proof.* See [2], Theorem 2.5.1.  $\square$

Let  $U \subseteq M$  be such that  $\xi|_U = \ker \alpha$  for some  $\alpha \in \Omega^1(U)$ . There exists a unique vector field  $R \in \mathfrak{X}(U)$  such that  $\alpha(R) = 1$  and  $R \lrcorner d\alpha = 0$ , called the Reeb vector field. If  $X \in \mathfrak{X}(U)$  is a complete vector field, then its flow  $\text{Fl}^X$  consists of contactomorphisms if and only if

$$(4.1) \quad \mathcal{L}_X \alpha = u\alpha$$

for some  $u \in C^\infty(U)$ . If we take any  $f \in C^\infty(U)$ , by nondegeneracy of  $d\alpha$  there exists  $X_f \in \mathfrak{X}(U)$  satisfying  $\alpha(X_f) = f$  and  $X_f \lrcorner d\alpha = df(R)\alpha - df$ . These conditions imply equality (4.1). On the other hand, if  $X$  satisfies (4.1), then it is of the form  $X_f$  for  $f = \alpha(X)$ .

For more information on contact manifolds see [2].

#### 4.2. Representations of $\text{Cont}_c(M, \xi)$ .

**Lemma 4.2.** *Let  $p \in M$  and let  $\phi: U \rightarrow H_n$  be a Darboux chart centered at  $p$ . Then there exist an open set  $V \subseteq H_n$ , a convex open neighborhood  $W$  of 0 in the Lie algebra of  $H_n$ , and for every  $x \in \exp[W]$  a contactomorphism  $\rho_x \in \text{Cont}_c(U, \xi|_U) \subseteq \text{Cont}_c(M, \xi)$  such that*

- (1)  $0 \in V \subseteq VV \subseteq \exp[W] \subseteq \overline{V \exp[W]} \subseteq \phi[U]$ ,
- (2)  $\phi \rho_x \phi^{-1}(y) = yx$  for all  $y \in V$ .

*Proof.* Existence of  $V$  and  $W$  satisfying (1) is obvious. Let  $x = \exp v$ , where  $v \in W$ . Then  $v$  extends to a left-invariant vector field  $X \in \mathfrak{X}(H_n)$ , and  $\text{Fl}_t^X = R_{\exp tv}$ , where  $R_y$  is the right multiplication by  $y$ . If  $f = h\alpha_0(X)$ , where  $h|_{V \exp[W]} = 1$  and  $\text{supp } h \subseteq \phi[U]$ , then  $X_f = X$  on  $V \exp[W]$ . The contactomorphism  $\rho_x = \phi^{-1} \text{Fl}_1^{X_f} \phi$  satisfies condition (2).  $\square$

**Corollary 4.3.** *The action of  $\text{Cont}_c(M, \xi)$  on  $M$  is  $k$ -transitive for all  $k \geq 1$ .*

**Lemma 4.4.** *Let  $\phi: U \rightarrow H_n$  be a Darboux chart. Then for every nontrivial  $\Pi_\mu^\theta$ -invariant  $\mathcal{H}_0 \leq L^2(M, \mu)$ , there exists  $f \in \mathcal{H}_0$  such that  $f \neq 0$  and  $\text{supp } f \subseteq U$ .*

*Proof.* Without loss of generality assume that  $0 \in U \subseteq H_n$  and  $\xi|_U = \ker \alpha_0$ . Let  $\delta_t(\bar{x}, \bar{y}, z) = (e^t \bar{x}, e^t \bar{y}, e^{2t} z)$  be the flow of the field  $X = (\bar{x}, \bar{y}, 2z)$ . We have  $\delta_t^* \alpha_0 = e^{2t} \alpha_0$ , so  $X = X_g$  for some function  $g \in C^\infty(H_n)$ .

There exist  $V = B(0, r) \subseteq \bar{V} \subseteq U$  and a function  $h$  supported in  $U$ , such that  $h|_V = g|_V$ . Let  $\psi_t = \text{Fl}_t^{X_h}$ . Then  $\psi_t|_V = \delta_t|_V$  for  $t < 0$ . Now, by transitivity of  $\text{Cont}_c(M, \xi)$ , we may take a nonzero  $f \in \mathcal{H}_0$  such that  $\text{supp } f \cap V \neq \emptyset$ . Since

$$(4.2) \quad \int_V |\Pi_\mu^\theta(\psi_t)f|^2 d\mu = \int_{\psi_{-t}[V]} |f|^2 d\mu \xrightarrow{t \rightarrow \infty} 0,$$

there exists  $t > 0$  such that  $f - \Pi_\mu^\theta(\psi_t)f$  satisfies the conclusion of the lemma.  $\square$

Now, fix a Darboux chart  $\phi: U \rightarrow H_n$  and a Lebesgue measure  $\mu$  on  $M$ , such that  $0 \in \phi[U]$  and  $\phi_*\mu$  is the standard Lebesgue measure on  $\phi[U] \subseteq \mathbb{R}^{2n+1}$ .

**Theorem 4.5.** *For every  $\theta \in \mathbb{R}$  the representation  $\Pi_\mu^\theta$  of  $\text{Cont}_c(M, \xi)$  on the space  $L^2(M, \mu)$  is irreducible.*

*Proof.* The proof is analogous to the proof of Theorem 3.2. Lemma 4.2 gives us  $V \subseteq U$  and contactomorphisms  $\rho_x$ , such that for  $f$  and  $g$  supported in  $V$  the matrix coefficient  $\langle f, \Pi_\mu^\theta(\rho_x)g \rangle$  is nonzero for some  $\rho_x$  because of Theorem 2.4.  $\square$

## REFERENCES

- [1] Ana Cannas da Silva. *Lectures on Symplectic Geometry*. Springer, 2001.
- [2] Hansjörg Geiges. *An Introduction to Contact Topology*. Cambridge University Press, 2008.
- [3] Lars Hörmander. *The Analysis of Linear Partial Differential Operators I*. Springer-Verlag, 1990.
- [4] R. S. Ismagilov. *Representations of Infinite-Dimensional Groups*. American Mathematical Society, 1996.
- [5] Dusa McDuff and Dietmar Salamon. *Introduction to Symplectic Topology*. Oxford University Press, 1998.
- [6] A. M. Vershik, I. M. Gel'fand, and M. I. Graev. Representations of the group of diffeomorphisms. In *Representation theory: selected papers*. Cambridge University Press, 1982.

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